BRITISH MATHEMATICAL OLYMPIAD

1973

TIME: 3 HOURS

PLEASE NOTE INVIGILATOR'S INSTRUCTIONS

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1. (i) Two fixed circles are touched by a variable circle at P and Q.

   Prove that PQ passes through one of two fixed points.

   (ii) State a true theorem about ellipses or if you like about conics in general of which (i) is a particular case.

2. 9 points are given in the interior of the unit square.

   Prove there exists a triangle of area $\leqslant \frac{1}{8}$ whose vertices are three of the points.

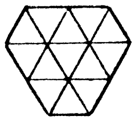
3. A curve consisting of the quarter-circle $x^2 + y^2 = r^2$, $x, y \geqslant 0$, together with the line segment $x = r$, $-h \leqslant y \leqslant 0$, is rotated about $x = 0$ to form a surface of revolution which is a hemisphere on a cylinder. A string is stretched tightly over the surface from the point on the curve $(r\sin\theta, r\cos\theta)$ to the point $(-r, -h)$ in the plane of the curve. Show that the string does not lie in a plane if $\tan\theta > \frac{r}{h}$.

   [You may assume spherical triangle formulae such as $\cos a = \cos b\cos c + \sin b\sin c\cos A$ or $\sin A\cot B = \sin c\cot b - \cos c\cos A$. In a spherical triangle the sides a, b, c are arcs of great circles and are measured by the angles they subtend at the centre of the sphere.]

P.T.O....

4. You have a large number of congruent triangular equilateral discs on a table and you want to fit n discs together to make a convex equiangular hexagon (i.e. one whose interior angles are each 120°).

Obviously n cannot be *any* positive integer. The smallest n is 6, the next smallest is 10 and the next 13. Determine conditions for possible n.



$n = 13$

5. There is an infinite set of positive integers of the form $2^n - 3$ with the property Q: no two members of the set have a common prime factor. An outline of a proof is as follows.

Suppose there is a finite set $S = (2^{m_1} - 3, 2^{m_2} - 3, \ldots\ldots\ldots 2^{m_k} - 3)$ with property Q and k members. Let the prime factors of these k numbers be $P_1, P_2 \ldots\ldots\ldots P_t$. Consider the number $N = 2^{(P_1 - 1)(P_2 - 1)\ldots\ldots(P_t - 1)} + 1$. By Fermat's theorem $a^{P-1} - 1 \equiv 1 \pmod{P}$ for every prime P that does not divide a. Hence $N - 3 \equiv -1 \pmod{P_r}$, $r = 1$ to t, and $N - 3$ may be added to S to give a larger set with property Q.

Give a properly expanded and reasoned proof that there is an infinite set of positive integers of the form $2^n - 7$ with property Q.

6. In answering general knowledge questions (framed so that each question is answered yes or no) the teacher's probability of being correct is α and a pupil's probability of being correct is β or γ according as the pupil is a boy or a girl.

The probability of a randomly chosen pupil agreeing with the teacher's answer is $\frac{1}{2}$.

Find the ratio of the numbers of boys to girls in the class.

7. The life-table issued by the Registrar-General of Draconia shows out of 10,000 live births the number (y) expected to be alive x years later. When $x = 60$, $y = 4{,}820$. When $x = 80$, $y = 3{,}205$. For $60 \leqslant x \leqslant 100$ the curve $y = Ax\,(100 - x) + \dfrac{B}{(x - 40)^2}$ fits the figures in the table very closely, A and B being constants.

Determine the life-expectancy (in years correct to one decimal place) of a Draconian aged 70.

N.B. At age 100 all Draconians are put to death.

8. Call $M_r = \begin{pmatrix} a_r & b_r \\ c_r & d_r \end{pmatrix}$ the companion matrix for the mapping $T_r : z \longrightarrow \dfrac{a_r z + b_r}{c_r z + d_r}$. Det $M_r \neq 0$.

(i) Prove that $M_1 M_2$ is the companion matrix for the mapping $T_1 T_2$.

(ii) Find conditions on a, b, c, d so that $T^4 = I$ but $T^2 \neq I$.

9. $$L_r = \begin{vmatrix} x & y & 1 \\ a + c\cos\theta_r & b + c\sin\theta_r & 1 \\ l + n\cos\theta_r & m + n\sin\theta_r & 1 \end{vmatrix}$$

Show that the lines $L_r = 0$, $r = 1, 2, 3$ are concurrent and find the co-ordinates of their concurrence.

10. Construct a detailed flow chart for a computer program to print out all positive integers up to 100 of the form $a^2 - b^2 - c^2$, where a, b, c are positive integers and $a \geqslant b + c$.

There is no need to print in ascending order or to avoid repetitions.

P.T.O.....

11. (i) Two uniform rough right circular cylinders A and B, with the same length, have radii and masses a, b and M, m respectively.

A rests with a generator in contact with a rough horizontal table. B rests on A, initially in unstable equilibrium, with its axis vertically above A's. Equilibrium is disturbed, B rolls on A and A rolls on the table. In the subsequent motion the plane containing the axes makes an angle θ with the vertical.

Draw diagrams showing angles, forces etc., for the period when there is no slipping. Write down equations which will give on elimination a differential equation for θ, stating the principles used. *Indicate* how the elimination could be done; you are not asked to do it.

(ii) Such a differential equation is, with $k = M/m$

$$\ddot{\theta}(4 + 2\cos\theta - 2\cos^2\theta + 9\tfrac{k}{2}) + \dot{\theta}^2 \sin\theta\,(2\cos\theta - 1)$$

$$= \frac{3g\,(1 + k)\sin\theta}{a + b}$$

Obtain $\dot{\theta}$ in terms of θ.

[Moment of inertia of a uniform cylinder about its axis is $\frac{1}{2}$ (mass) (radius)^2]

Mathematical Association
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